SOAR Research Proposal – Summer 2016
Edge and Vertex Failure Games on Graphs

Faculty: Nathan Shank, Associate Professor, Mathematics and Computer Science Department
Dates: May 31, 2016 - August 5, 2016 (10 weeks)
Student: Devon Vukovich, Junior Mathematics Major
Graduation: Spring 2017
Title: Edge and Vertex Failure Games on Graphs

Description of the Project

Network reliability and vulnerability has become a very important issues in many different areas including social structure, epidemiology, biological applications, and communication networks. Companies want to build cost effective networks while hackers want to exploit their weaknesses. Doctors want to protect patients while diseases want to spread. Information needs to be shared with individuals while groups want to keep information from spreading. In mathematics we model networks as a graph consisting of a collection of vertices and a set of edges between vertices. The vertices can represent people, computers, or communication towers while the edges between two vertices presents a specific relationship between two vertices. For example, if two people have had any sexual contact or if two computers share information, or if there is communication between two towers.

A typical network reliability measure is built using two factors; 1) understanding what is prone to failure or attack, and 2) what are the requirements for a network to "fail" or be in a failure state. Network reliability models have only been studied for the past 50 years. The early models considered edge failures and a failure state consisted of disconnecting the graph into more than one component. The other objects prone to failure include vertices or mixed (vertices and edges). In any of these models we are trying to find the minimum number of failures in order to produce a failure state. In other words, how quickly could the network break, hence a larger network reliability parameter implies a more reliable network.

In the past 20 years several important variations have been considered. Particular attention has been focused on the *Component Edge (Vertex) Connectivity Parameter* of a network. This model considered edge (vertex) failures, but a network is in a failure state only if the remaining components are all smaller than some threshold. Thus the network is operational if there are enough vertices connected. A 2008 SOAR student Phillip Gaudreau and I published a paper on extremal graphs based on this model. Phillip is now finishing his Ph.D. work at Drexel in Graph Theory.

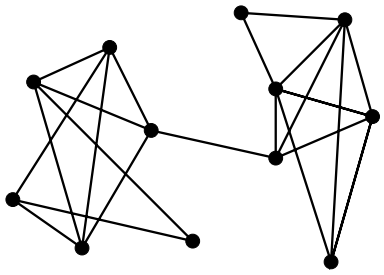

Figure 1: *An example graph*

In this current SOAR proposal, we want to look at a new model of reliability which includes the idea of immunity. Consider a network of people and edges representing contact with allows spread of a disease. If a disease starts with a particular person and spreads to their contacts, however other people can become immune to the disease because of a vaccine, is it possible to stop the outbreak of the disease? How fast must we vaccinate people to stop the spread of the

disease? Although answering these questions is a lofty goal for a summer SOAR project, we plan to follow the classical steps in conducting graph theory research by simplifying the problem and then expanding it to make it more difficult. We will first start by considering extremely simple graph structures including path graphs, cycles, or lattice structures.

This problem is particularly interesting because of the overlap with Game Theory. The same problem outlined above can be thought of as a game between two players. Consider the graph structure as a "board." One player colors vertices red (infected) and will be considered the "attacker" and the other player colors vertices black (immunity) will be considered the "defender". Players alternate turns coloring vertices however you can only color a vertex which shares an edge with a vertex of the same color. This models the spread of immunity or the spread of the disease. Is there a "winning" strategy for either player? This will depend partially on how we define a "winner."

We hope to look at several variants of this project including varying the graph structure, edge and vertex properties, and the definition of "winner." Another extremely interesting example would be to allow the defender to quarantine an individual, thus deleting all the edges from that vertex. Therefore if a particular country has a high infection rate for a particular disease, to stop the spread of the disease is it more useful to develop a vaccine or to restrict travel to and from that country?

The main goal of this project is to have Devon understand and experience what it means to do math research. As a transfer student, Devon participated in the MCS^2 summer program last summer and worked on a research project for 20 hours per week for 10 weeks. This previous research experience gave him a glimpse of the first stages of mathematical research, particularly refining problems and developing conjectures. This summer SOAR project will allow Devon to engage in all aspects of mathematical research including refining a problem, developing conjectures, proving results, preparing the results for submission to peer reviewed journals, and presentation to mathematical and general audiences.

For me, this project is a natural progression of my current research agenda. It combines Graph Theory, Game Theory, and Combinatorics. I'm extremely excited that the project was developed jointly with Devon and he shares the same level of excitement for the project as I do. Devon completed a Graph Theory course in the Fall of 2015 and is currently taking an independent study in Combinatorics which has included several topics in Game Theory. Therefore Devon will not need to do much background reading in order to understand previous literature and engage in the project quickly. Devon is an extremely mathematically mature, talented, and focused student. He has the potential to be one of the best mathematics students Moravian has produced in the past 10 years! He plans to do Honors next year and take an LVAIC math course at Lehigh. His goal is to attend graduate school in mathematics and I have exceptionally high expectations for him. The summer SOAR program will give Devon the much needed experience and credentials to allow him to achieve his goals.

Roles and Responsibilities Devon and I will share responsibilities on all aspects of the project, although we may devote varying amounts of time to each part.

1. Literature Review (Both): The project will begin with some background reading on different reliability measures. Devon will be responsible finding and summarizing any material that

has previously been studied on this topic. We have already begin collecting background material during our independent study.

2. Develop Conjectures (Both): Devon and I will start to develop questions and conjectures based on the different game structures. We will follow the pattern in previous literature by working with simple graph classes. We will likely have to adjust our definitions and structure to find the appropriate level of difficulty. This is a valuable step for a student to understand how to conduct mathematical research. Often in mathematical research the original problem will need to undergo several changes and possible simplifications until it becomes something solvable.
3. Prove Conjectures (Both): Devon and I will spend the bulk of our time working on proofs for our conjectures. This may involve refining the problem in order to get provable results.
4. Written Results (Both): We will write up our results into a publishable paper for submission. Devon and I will work on the outline together, however Devon will take the lead on preparing the first draft.

Timetable

- 1 weeks: Background reading
- 2 weeks: Developing conjectures based on readings and current work
- 5 weeks: Attempt to prove conjectures and making appropriate adjustments
- 2 weeks: Write up results and prepare presentations

Engagement in Discipline-Appropriate Scholarly Research

This project will follow the general outline of pure mathematical research. This includes gaining background information, developing sound definitions and rules, analyzing or proving results based on these definitions and rules, and communicating (both written and verbally) results to different audiences. Devon will have a major role in all phases of the project.

Contribution to the Discipline and Opportunities to Share Work

This project will contribute knowledge to the general understanding of Game Theory and Network Reliability which engages many different disciplines. I expect at least one peer reviewed journal paper and several presentations as a result of this SOAR project. Devon will share his work at the regional *Eastern Pennsylvania and Delaware* section meeting of the *Mathematical Association of America*, Moravian Scholars Day, and the annual *Moravian Student Mathematics Conference*.

Summary of Benefits

The benefits of this project for the student include developing the students understanding of mathematical research, strengthening the students credentials for graduate school, and improving upon the level of undergraduate education for the students. For me, the benefits include developing skills in mentoring undergraduates, contributing significant results to the general mathematics field, and engaging students in several areas of my research at the same time. For the department and the college the benefits include increasing mathematical awareness on campus and enhancing the scholarly atmosphere.

SOAR Student Statement – Summer 2016
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Housing: YES

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I have always had a deep interest in mathematics. To those who knew me as a child, this came as no surprise. I have been solving puzzles since I could walk, and my parents have amusing anecdotes to prove it. Mathematics brings me joy in the way it marries simplicity and complexity. Mathematics I am unfamiliar with is intimidating gibberish, but the moment you learn how to approach these monstrosities is the moment the complex becomes simple. I live for that "'Aha"' moment, that moment that everything clicks, and I feel enlightened in a way I have never felt before. Learning and exploring mathematics constantly provides these moments. Unfortunately, I lost my taste for academics in high school. After some time traveling and maturing, I realized that mathematics is my passion, and I would not be happy unless I pursued mathematics professionally. Because I had spent some time outside of school, I decided to attend Lehigh Carbon Community College to build an academic record that reflected my talents accurately. I received an A.S. in Physical Science and nearly completed a second A.S. in Mathematics. The completion of my two year degree meant I needed to transfer to continue my growth and studies. I was accepted to the MCS^2 program here at Moravian College. This allowed me to continue my education beginning with a summer class and a research opportunity. I have spent almost a year at Moravian College now, and that time has seen me grow and develop as a student and mathematician in ways I never expected. This is brought not just by the classes I have taken, but by the opportunities that are available to learn outside the classroom.

It is difficult to determine exactly where my interest in these topics began. I was first exposed to the concepts of Combinatorics and Game Theory by the television show Numbers, but I had very little understanding of the mathematics behind said concepts. I was first exposed to Graph Theory in Discrete Mathematics at Lehigh Carbon Community College. I first started understanding Graph Theory and Combinatorics in Discrete Mathematics here at Moravian College. I started learning Graph Theory in a special topics class offered here, and I started learning Combinatorics and Game Theory in an independent study also done here. Since my participation in MCS^2 last summer, I have been actively working towards being able to complete a SOAR project. I have been building the background and foundational knowledge necessary to perform original research in the fields that I find most intriguing. I am drawn towards a SOAR project because it allows me to work closely with my adviser on a project that I help design. Dr. Shank has been my adviser and mentor since I first came to Moravian College. He understands my goals and wants to help guide me towards achieving them. Conducting a SOAR project with him is a natural and logical

progression of my education that will allow him to instruct me in developing the skills I will need as a research mathematician.

My primary education goal is to pursue graduate studies culminating in a Ph.D. Conducting independent research is a large part of achieving these goals. My intention is to gain exposure to this process as soon as possible. Optimally this begins with the completion of a SOAR project, and then transitions into a Senior Honors Project. One large criteria I had for developing this project was to examine something with enough depth that I could continue this research for the said Honors Project I intend to begin Fall 2016. Both SOAR and a Senior Honors Project will not only expose me to the skills I will need as a research mathematician, but give me a cohesive project to include in my portfolio of work. This will strengthen my applications to graduate studies, and strengthen my abilities to allow me to be the most successful student I can be.

After achieving my education goals, I intend to pursue a professorship and a career in mathematical research at a college or university. This long term goal allows me to continue to learn and explore mathematics which is my joy, but also will allow me to do it in a way that gives back to the community that helped me achieve my goals. Throughout my education, I have worked as a tutor in mathematics. The one thing I learned as a tutor is that education is the most rewarding when it is a process of give and take. A professorship would allow me to act in the same role that Dr. Shank has acted for me. I would be able to help encourage and develop undergraduates. I would be able to mentor them in projects like SOAR. I want this to be a part of my career as much as I want mathematics to be a part of my career. Therefore, participation in SOAR will help me build skills in not only research, but mentoring that will be critical to my success professionally.

The outcomes I expect for this project are to write a submission to a peer-reviewed journal, and prepare presentations for mathematical and general audiences. Further outcomes I expect are to increase my comfort in writing and using mathematical software (Tex, Maple, etc.), explore and better understand the mathematical research process (performing background research, developing definitions and conjectures, analyzing and proving results), and to practice educational independence.