Disjoint path covers of hypertori

Let G be a simple graph, and let V be the vertex set of G. Let $S = \{s_1, s_2, \ldots, s_k\}$ and $T = \{t_1, t_2, \ldots, t_k\}$ be disjoint subsets of V. A *paired disjoint k-path cover* of G is a subgraph of G consisting of paths $P_1, P_2, \ldots, P_k$ such that

- each path P_i has endpoints s_i and t_i, and
- the vertex sets of the paths partition V.

We will abbreviate the term "paired disjoint k-path cover" as "k-path cover". If G has a k-path cover for every choice of S and T, then G is said to be *paired k-to-k disjoint path coverable*. If $k = 1$, then G is also said to be *Hamiltonian connected*.

Let G be a bipartite graph with partite sets V_1 and V_2. Let $|V_1| - |V_2| = \delta$. We say that $S \cup T$ is *balanced* if $|(S \cup T) \cap V_1| - |(S \cup T) \cap V_2| = 2\delta$. Note that $S \cup T$ being balanced is a necessary condition for the bipartite graph G to have a k-path cover with endpoints S and T. If G has a k-path cover for every choice of S and T such that $S \cup T$ is balanced, then G is said to be *balanced paired k-to-k disjoint path coverable*. If $k = 1$, then G is also said to be *Hamiltonian laceable*.

Let n be a positive integer, and let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be an n-tuple of integers such that $d_i \geq 2$ for all i. A hypertorus $Q_n^{\mathbf{d}}$ is a simple graph defined on the vertex set $\{(v_1, v_2, \ldots, v_n) : 0 \leq v_i \leq d_i - 1 \text{ for all } i\}$, and it has edges between $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ if and only if there exists a unique i such that $|u_i - v_i| = 1$ or $d_i - 1$, and for all $j \neq i$, $u_j = v_j$. A two-dimensional hypertorus $Q_2^{\mathbf{d}}$ is simply a torus, and if $d_1 = d_2 = \cdots = d_n = 2$, then $Q_n^{\mathbf{d}}$ is an n-dimensional hypercube.**r**A

Gregor and Dvo**v**rák [1] proved that when $n \geq 3$, the n-dimensional hypercube is balanced paired $(\lceil \frac{n}{2} \rceil - 1)$-to-$(\lceil \frac{n}{2} \rceil - 1)$ disjoint path coverable. Kronenthal and Wong [2] proved that if $d_1 \geq 3$ and $d_2 \geq 3$, then $Q_2^{\mathbf{d}}$ is balanced paired 2-to-2 disjoint path coverable if both d_i are even, and is paired 2-to-2 disjoint path coverable otherwise. The main conjecture is given by the following.

Conjecture 0.1. *Let $n \geq 2$ be an integer, and let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be an n-tuple of integers such that $d_i \geq 3$ for all $i = 1, 2, \ldots, n$. If $d_1, d_2, \ldots, d_n$ are not all even, then $Q_n^{\mathbf{d}}$ is paired n-to-n disjoint path coverable. Otherwise, $Q_n^{\mathbf{d}}$ is balanced paired n-to-n disjoint path coverable.*

This conjecture is the strongest possible in the sense that $Q_n^{\mathbf{d}}$ cannot be $(n+1)$-to-$(n+1)$ disjoint path coverable. This is because if we pick the endpoints $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n$ to be neighbors of $\mathbf{s}_{n+1}$, then there is obviously no path that joins $\mathbf{s}_{n+1}$ with $\mathbf{t}_{n+1}$ since every vertex in $Q_n^{\mathbf{d}}$ has degree $2n$.

For $n \geq 3$, Kronenthal and Wong [2] showed that the truth of Conjecture 0.1 hinges on the base cases when d_i are small.

Theorem 0.2. *Let $n \geq 2$ be an integer, and let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be an n-tuple of integers such that $d_i \geq 3$ for all $i = 1, 2, \ldots, n$.*

1. *Suppose* $d_1, d_2, \ldots, d_n$ *are not all even. If* $Q_n^{\mathbf{d}}$ *is paired* n*-to-*n *disjoint path coverable for all* $\mathbf{d}$ *such that* $3 \leq d_1, d_2, \ldots, d_n \leq 4n$*, then* $Q_n^{\mathbf{d}}$ *is paired* n*-to-*n *disjoint path coverable for all* $\mathbf{d}$*.*

2. *Suppose* $d_1, d_2, \ldots, d_n$ *are all even. If* $Q_n^{\mathbf{d}}$ *is balanced paired* n*-to-*n *disjoint path coverable for all* $\mathbf{d}$ *such that* $3 \leq d_1, d_2, \ldots, d_n \leq 4n$*, then* $Q_n^{\mathbf{d}}$ *is balanced paired* n*-to-*n *disjoint path coverable for all* $\mathbf{d}$*.*

This project is suitable for an REU project on experimental mathematics, since the base cases specified by Theorem 0.2 can be checked with efficient computer programming.

References

[1] P. Gregor and T. Dvo**v**rák, Path partitions of hypercubes, *Information Processing Letters* **108** (2008), 402–406.

[2] B. Kronenthal and T. W. H. Wong, Paired many-to-many disjoint path covers of hypertori, *Discrete Applied Mathematics* **218** (2017), 14–20.