Constructing matrices with prescribed number of bases

Let A be a matrix of size $r \times n$ over a field. Assume that A has full row rank, so we have $0 < r \leq n$ as a consequence. Let b be the number of invertible $r \times r$ submatrices of A. What are the possible values of b? From basic linear algebra and simple counting, we know that b must be between 1 and $\binom{n}{r}$ inclusively.

A more interesting and challenging question is whether every such b is attainable. In other words, if we are given a triple of positive integers (n, r, b), where $0 < r \leq n$ and $1 \leq b \leq \binom{n}{r}$, can we always build a matrix A of size $r \times n$ over a field, such that the number of invertible $r \times r$ submatrices is exactly b? We denote such a matrix as an (n, r, b)-*matrix*.

In fact, the proposed question is a restricted version of what Welsh [2] asked: Given a triple of positive integers (n, r, b), where $0 < r \leq n$ and $1 \leq b \leq \binom{n}{r}$, does there always exists a matroid with n elements, rank r, and exactly b bases? Such a matroid would be called an (n, r, b)-*matroid*.

Unfortunately, the answer to this question is known to be negative. According to Mayhew and Royle [1], Anna de Mier answered Welsh's question negatively by proving that an (n, r, b)-matroid does not exist when $(n, r, b) = (6, 3, 11)$. This implies that a $(6, 3, 11)$-matrix does not exist. However, Mayhew and Royle conjectured that this is the lone counterexample.

Conjecture 0.1 ([1])**.** *An (n, r, b)-matroid exists for all $0 < r \leq n$ and $1 \leq b \leq \binom{n}{r}$ except $(n, r, b) = (6, 3, 11)$.*

We strengthen Conjecture 1.1 as follows.

Conjecture 0.2. *An (n, r, b)-matrix exists for all $0 < r \leq n$ and $1 \leq b \leq \binom{n}{r}$ except $(n, r, b) = (6, 3, 11)$.*

We have shown that Conjecture 1.2 holds for all $1 \leq b \leq \binom{r+3}{r}$, except for $(n, r, b) = (6, 3, 11)$. We also proved that Conjecture 1.2 holds when r and b are large with respect to k. We strengthen this result into the following theorem.

Theorem 0.3. *For each fixed $k \geq 3$, there exists $R \in \mathbb{N}$ such that for all $r \geq R$, $(r + k, r, b)$-matrices exist for all $1 \leq b \leq \binom{r+k}{r}$.*

The proof techniques involved were induction arguments, with base cases taken care of by computer programming.

References

[1] D. Mayhew and G. F. Royle, Matroids with nine elements, *J. Combinatorial Theory Ser. B* **98** (2008), 415–431.

[2] D. Welsh, Combinatorial problems in matroid theory, *Combinatorial Math. and its Applications (Proc. Conf. Oxford 1969)*, Academic Press, London, 1971.