SOAR Research Proposal - Summer 2015
Quasi-Crowns

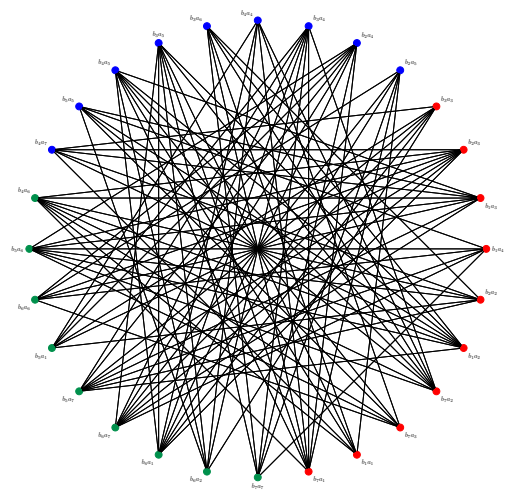
Project Information

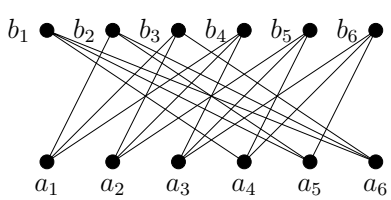
Title: Quasi-Crowns
Faculty Mentor: Dr. Shannon Talbott, Assistant Professor, Department of Mathematics and Computer Science
Student: Alexis Thiel
Start Date: June 1, 2015
Length of Project: 10 weeks

1 Project Description

This project considers a problem in Graph Theory, where a graph is a set of points called vertices with a set of edges that connect those vertices. One question that has motivated research for well over a century is the challenge of coloring a graph; that is, how many colors must one use to color vertices so that any two vertices that are connected by an edge are not the same color? (See the figure below for an example.)

The origin of the problem stems from addressing the question of the minimal number of colors needed to color countries in a map so that bordering countries are colored with different colors; see [1, 2, 3, 4, 6, 7, 8, 9]. Moreover, the graph coloring problem, also referred to as the chromatic number, is one of Karp's twenty-one NP-complete problems listed in his 1972 paper; see [5]. The chromatic number problem continues to motivate the research undertaken by current generations of mathematicians.





Crown $\mathbb{S}_4^2$

Dr. Talbott and her collaborators classified an infinite family of graphs, which arise from a set called a crown, for which we know an upper bound on the chromatic number. (See the figure above for a small example of a crown.) A crown is a graph with elements arranged in a top row and bottom row with an equal number of elements in each row. The elements that make up the top row are called maximal elements and the elements that make up the bottom row are called minimal elements. The example on the left is the crown $\mathbb{S}_4^2$, where each maximal element "misses" 3 minimal elements, the one directly below and the two to the right of that, and each maximal element "hits" the remaining 3 minimal elements. The number of "misses" is one more than the superscript of the name of the crown and the number of "hits" is one less than the subscript of the name of the crown.

The sum of the subscript and superscript must equal the number of elements in each row. A pair of elements is called a comparable pair if they are connected by an edge and are called an incomparable pair if they are not connected. For example, the maximal element b_1 is comparable with a_4, a_5, a_6 and is incomparable with a_1, a_2, a_3 and the other 5 maximal elements.

In this project, we would like to alter the definition of a crown and then explore resulting properties of the graph and its chromatic number. There are varied ways to alter the definition of a crown such as allowing the maximal elements and the minimal elements to be sets of different sizes. We then must explore various ways to describe incomparable pairs, i.e. paris of elements that are not connected by an edge. We would need to compute many examples in order understand how this changes both the graph and the chromatic number. Depending upon time, we might explore multiple ways to alter the definition.

2 Roles and Responsibilities

We begin with working through a few papers of background material necessary to understanding the project, most likely beginning with Dr. Talbott's most recent paper and then branching out to other references as necessary. Dr. Talbott will be responsible for using her expertise to fill in gaps in knowledge. We will both be responsible for considering interesting ways to alter the definition of a crown, and Alexis will be responsible for working out examples to help illustrate concepts in our definitions and conjectures. This project could result in a publication and thus will require us to write up our results in appropriate manner, which will require the use of a mathematical editor called LaTex. Alexis will be assisted by Dr. Talbott in learning and using this editor. Dr. Talbott will aid in the writing process by editing and putting ideas into an appropriate context to the history of the problem as well as current research.

2.1 Timeline of Milestones

- 2-3 weeks: Initial background work
- 5-6 weeks: Develop various definitions, work out examples and create conjectures, prove conjectures
- 1-2 weeks: Write up results and edit for concise wording and clarity

3 Discipline-Appropriate Scholarly Research

Alexis will be involved in every step of the mathematical research process and so will have the opportunity to learn what is required to successfully complete a mathematical research project. This process includes the development of new ideas to explore, computing examples to illustrate mathematical properties, creating conjectures about mathematical properties, and trying to prove these conjectures.

4 Contributions to the Discipline

There are many open questions to consider when discussing both the problem of coloring a graph as well as regards the infinite family of graphs known as crowns. We propose to begin answering some of these open questions with this project.

This work could result in publication, and so Alexis would be sharing the results with the mathematical community in this manner. There are other opportunities to share the research such as at conferences like MathFest (a conference held at the end of the summer by the MAA-Mathematical Association of America) where Alexis could present her work to mathematicians from across the country, the Moravian College Undergrad Mathematics conference where she could present her work to an audience of mostly undergraduate students from Pennsylvania, and the Math Society's weekly Epsilon talks where she could present her work to Moravian college.

5 Budget Items

One of Dr. Talbott's collaborators has been writing programs in Mathematica, software that aids in computation for examples of crowns and their resulting graphs. Moravian college has Maple, computational software, but the collaborator tasked with writing code does not have access to Maple. This collaborator is at the United States Military Academy.

- We request $100 for a student desktop license to Mathematica, software that can be used for computation when exploring examples.
- We request $400 for Alexis to visit to the United States Military Academy to discuss the creation of a program to use for computation of examples of quasi-crowns ($205 for mileage, $150 for accommodations, and $45 per diem).

References

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[3] P. J. Heawood, *Map-colour theorem*, Proc. London Math. Soc. (2) **51** (1949), 161–175. MR 0030735 (11,43d)

[4] L. Heffter, *Ueber das Problem der Nachbargebiete*, Math. Ann. **38** (1891), no. 4, 477–508. MR 1510685

[5] Richard M. Karp, *Reducibility among combinatorial problems*, Complexity of computer computations (Proc. Sympos., IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y., 1972), Plenum, New York, 1972, pp. 85–103. MR 0378476 (51 #14644)

[6] A. F. Möbius, *Uber die bestimmung des inhaltes eines polyeders*, vol. 17, Ber. K. Sächs. Ges. Wiss. Leipzig Math.–Phys. Cl., 1865, also Werke, vol. 2, pp.473–512.

[7] Gerhard Ringel, *Map color theorem*, Springer-Verlag, New York-Heidelberg, 1974, Die Grundlehren der mathematischen Wissenschaften, Band 209. MR 0349461 (50 #1955)

[8] Saul Stahl, *The other map coloring theorem*, Math. Mag. **58** (1985), no. 3, 131–145. MR 789569 (86g:05040)

[9] J. W. T. Youngs, *The Heawood map coloring conjecture*, Graph Theory and Theoretical Physics, Academic Press, London, 1967, pp. 313–354. MR 0236059 (38 #4357)